Question 1

t (minutes)	0	4	9	15	20
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is $55°\text{F}$. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

(a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

(b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

(c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.

(d) For $20 \le t \le 25$, the function W that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t}\cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

(a) $W'(12) \approx \frac{W(15) - W(9)}{15 - 9} = \frac{67.9 - 61.8}{6}$
$= 1.017$ (or 1.016)

The water temperature is increasing at a rate of approximately $1.017\ °\text{F}$ per minute at time $t = 12$ minutes.

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{interpretation with units} \end{cases}$

(b) $\int_0^{20} W'(t)\,dt = W(20) - W(0) = 71.0 - 55.0 = 16$

The water has warmed by $16\ °\text{F}$ over the interval from $t = 0$ to $t = 20$ minutes.

$2 : \begin{cases} 1 : \text{value} \\ 1 : \text{interpretation with units} \end{cases}$

(c)
$$\frac{1}{20}\int_0^{20} W(t)\,dt \approx \frac{1}{20}(4 \cdot W(0) + 5 \cdot W(4) + 6 \cdot W(9) + 5 \cdot W(15))$$
$$= \frac{1}{20}(4 \cdot 55.0 + 5 \cdot 57.1 + 6 \cdot 61.8 + 5 \cdot 67.9)$$
$$= \frac{1}{20} \cdot 1215.8 = 60.79$$

This approximation is an underestimate, because a left Riemann sum is used and the function W is strictly increasing.

$3 : \begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{approximation} \\ 1 : \text{underestimate with reason} \end{cases}$

(d)
$$W(25) = 71.0 + \int_{20}^{25} W'(t)\,dt$$
$$= 71.0 + 2.043155 = 73.043$$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

1A₁

t (minutes)	0	4	9	15	20
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

1. The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is 55°F. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

(a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

$W'(12) \approx \frac{67.9-61.8}{15-9} = \boxed{1.0167\ °F/min}$

At $t=12$, the temperature of the water in the tub is increasing at the rate of 1.067 °F/min.

(b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

$\int_0^{20} W'(t)dt = W(20) - W(0) = 71.0 - 55.0 = \boxed{16°F}$

$\int_0^{20} W'(t)dt$ is the difference in temperature in °F of the water in the tub at $t = 20$ and $t = 0$.

Unauthorized copying or reuse of

Continue problem 1 on page 5.

Do not write beyond this border.

1A₂

(c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.

$\int_0^{20} W(t)\,dt \approx 4(55.0) + 5(57.1) + 6(61.8) + 5(67.9)$
$= 1215.8$

$\frac{1}{20}\int_0^{20} W(t)\,dt = \boxed{60.79\,°F}$

As the function W(t) is strictly increasing, the approximation rectangles of the left Riemann sum fall below the curve. Thus the approximation is an underestimate.

(d) For $20 \le t \le 25$, the function W that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t}\cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

$\int_{20}^{25} W'(t) = W(25) - W(20) = 2.043$

$W(25) - 71.0 = 2.043$

$W(25) = \boxed{73.0432\,°F}$

GO ON TO THE NEXT PAGE.

1B,

t (minutes)	0	4	9	15	20
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

1. The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is 55°F. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

   (a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

$$W'(12) = \frac{W(15) - W(9)}{15 - 9} = \frac{67.9 - 61.8}{6} = \frac{6.1}{6} = 1.016 \frac{°F}{min}$$

At time t=12, the water is being heated at a rate of 1.016 °F/min.

   (b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

$$\int_0^{20} W'(t)\,dt = W(20) - W(0)$$
$$= 71.0 - 55.0 = 16.0 \text{ degrees Fahrenheit}$$

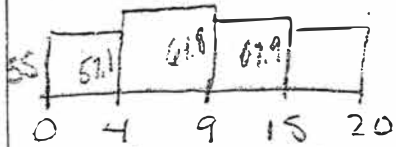
$\int_0^{20} W'(t)\,dt$ represents the change in water temperature in degrees Fahrenheit from time t=0 to t=20

Continue problem 1 on page 5.

Do not write beyond this border.

1B2

(c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.



$\frac{1}{20}\left[55(4-0)+57.1(9-4)+61.8(15-9)+67.9(20-15)\right]$

$\frac{1}{20}\left[55(4)+57.1(5)+61.8(6)+67.9(5)\right]$

$\frac{1}{20}\left[1215.8\right] = 60.79$

This approximation underestimates the average temp over the 20 minutes because $W''(t) > 0$. Concave up underestimates

(d) For $20 \le t \le 25$, the function W that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t}\cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

$W'(25) = 0.4\sqrt{25}\cos(.06(25))$
$= 0.141$

$y - 71.0 = 0.141(x-20)$
$y = 0.141x - 2.82 + 71$
$y = 0.141x + 68.18$

$W(25) = 0.141(25) + 68.18$
$= 71.705$

GO ON TO THE NEXT PAGE.

t (minutes)	0	4	9	15	20
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

1. The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is 55°F. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

(a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

$$W'(12) \approx \frac{W(t)_2 - W(t)_1}{t_2 - t_1}$$

$$\approx \frac{W(9) - W(4)}{9-4}$$

$$\approx \frac{61.8 - 57.1}{9-4} \approx \frac{4.7}{5} \approx .94 \text{ degrees/min}$$

(b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

$$\int_0^{20} W'(t)\,dt = W(t)\Big]_0^{20}$$

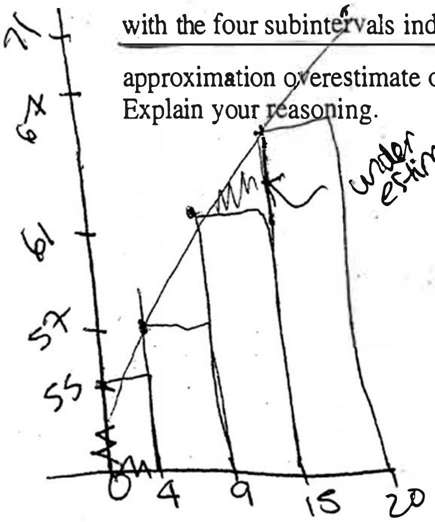
$$W(20) - W(0)$$

$$71 - 55 = 16 \text{ degrees Fahrenheit}$$

Do not write beyond this border.

1C2

(c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.



$(4-0)(55) + (9-4)(57.1) + (15-9)(61.8) + (20-15)(67.9)$

$220 + 285.5 + 370.8 + 339.5$

$\frac{1215.8}{20}$ → 60.79 avg temp

☆ This approximation underestimates the average temperature as seen in the graph above. Since the height of the rectangles are being cut short, the estimate is smaller than the actual.

(d) For $20 \le t \le 25$, the function W that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t}\cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

20.43154699

20°F

Do not write beyond this border.

Question 1

Overview

This problem involved a function W that models the temperature, in degrees Fahrenheit, of water in a tub. Values of $W(t)$ at selected times between $t = 0$ and $t = 20$ minutes are given in a table. Part (a) asked students for an approximation to the derivative of the function W at time $t = 12$ and for an interpretation of the answer. Students should have recognized this derivative as the rate at which the temperature of the water in the tub is increasing at time $t = 12,$ in degrees Fahrenheit per minute. Because $t = 12$ falls between the values presented in the table, students should have constructed a difference quotient using the temperature values across the smallest time interval containing $t = 12$ that is supported by the table. Part (b) asked students to evaluate the definite integral $\int_0^{20} W'(t)\,dt$ and to interpret the meaning of this definite integral. Students should have applied the Fundamental Theorem of Calculus and used values from the table to compute $W(20) - W(0).$ Students should have recognized this as the total change in the temperature of the water, in degrees Fahrenheit, over the 20-minute time interval. In part (c) students were given the expression for computing the average temperature of the water over the 20-minute time period and were asked to use a left Riemann sum with the four intervals given by the table to obtain a numerical approximation for this value. Students were asked whether this approximation overestimates or underestimates the actual average temperature. Students should have recognized that for a strictly increasing function, the left Riemann sum will underestimate the true value of a definite integral. In part (d) students were given the symbolic first derivative $W'(t)$ of the function W that models the temperature of the water over the interval $20 \le t \le 25,$ and were asked to use this expression to determine the temperature of the water at time $t = 25.$ This temperature is computed using the expression $W(25) = W(20) + \int_{20}^{25} W'(t)\,dt,$ where $W(20) = 71$ is given in the table.

Sample: 1A
Score: 9

The student earned all 9 points.

Sample: 1B
Score: 6

The student earned 6 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and no points in part (d). In parts (a) and (b) the student's work is correct. In part (c) the student earned the left Riemann sum and approximation points. The student does not give a correct reason for "underestimates," so the last point in part (c) was not earned. In part (d) the student's work is incorrect.

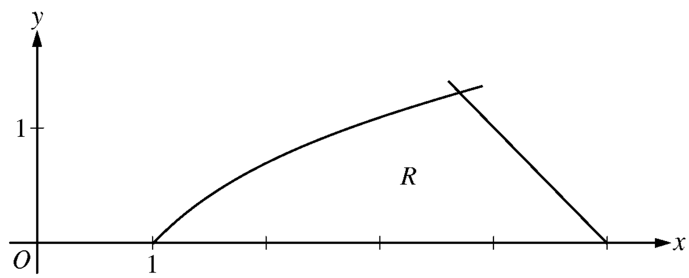
Sample: 1C
Score: 3

The student earned 3 points: no points in part (a), 1 point in part (b), 2 points in part (c), and no points in part (d). In parts (a) and (d) the student's work is incorrect. In part (b) the student earned the value point. In part (c) the student earned the left Riemann sum and approximation points. The student does not give a correct reason for "underestimates," so the last point in part (c) was not earned.

AP® CALCULUS AB
2012 SCORING GUIDELINES

Question 2

Let R be the region in the first quadrant bounded by the x-axis and the graphs of $y = \ln x$ and $y = 5 - x$, as shown in the figure above.



(a) Find the area of R.

(b) Region R is the base of a solid. For the solid, each cross section perpendicular to the x-axis is a square. Write, but do not evaluate, an expression involving one or more integrals that gives the volume of the solid.

(c) The horizontal line $y = k$ divides R into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

$\ln x = 5 - x \quad \Rightarrow \quad x = 3.69344$

Therefore, the graphs of $y = \ln x$ and $y = 5 - x$ intersect in the first quadrant at the point $(A, B) = (3.69344, 1.30656)$.

(a) Area $= \int_0^B \left(5 - y - e^y\right) dy$
$= 2.986$ (or 2.985)

OR

Area $= \int_1^A \ln x \, dx + \int_A^5 (5 - x) \, dx$
$= 2.986$ (or 2.985)

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$

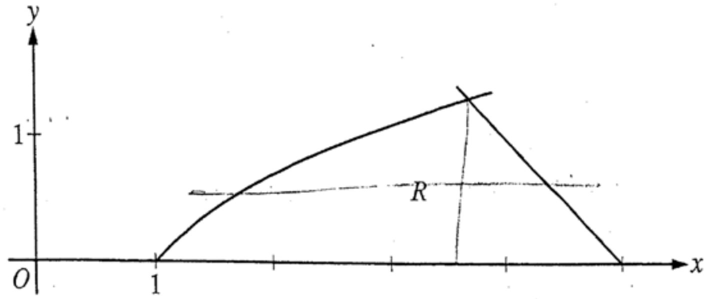
(b) Volume $= \int_1^A (\ln x)^2 \, dx + \int_A^5 (5 - x)^2 \, dx$

$3 : \begin{cases} 2 : \text{integrands} \\ 1 : \text{expression for total volume} \end{cases}$

(c) $\int_0^k \left(5 - y - e^y\right) dy = \frac{1}{2} \cdot 2.986 \left(\text{or } \frac{1}{2} \cdot 2.985\right)$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{equation} \end{cases}$

 2A₁



2. Let R be the region in the first quadrant bounded by the x-axis and the graphs of $y = \ln x$ and $y = 5 - x$, as shown in the figure above.

$e^y = x \quad x = 5 - y$

(a) Find the area of R.

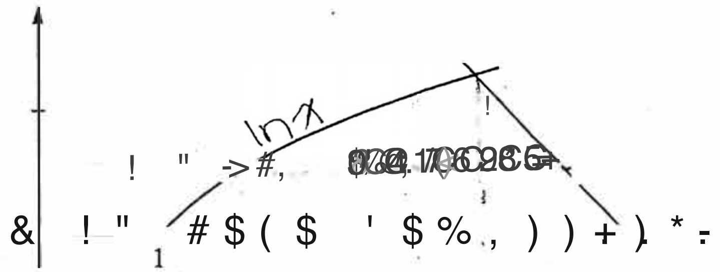
$$A = \int_0^{1.307} (5 - y - e^y)\,dy$$

$$= \boxed{2.986}$$

Do not write beyond this border.

Continue problem 2 on page 7.

Do not write beyond this border.



%# , 2GR R/2R72R258@9R6R2F9C1C1B . 1C=G@>121R PR72R. G9F.R>1R72R5C A6FR@RQR(R;?R . >1RQR) R&R . F
F7@V>9R62R5I C2R/ @2"

. +9>1R62R C2 R@4R #

~~Area = $\int_1^5 (\ln x) - (5-x)\,dx = 3.953$ units^2~~

Area = $\int_1^{3.693} (\ln x)\,dx$

Area = 2.132 +

Do not write beyond this border.

1 , 47:GFe ":F0X84e10S4eG5e0eFG?,3&a-G4eX84eSG>3!e4028e2LGSSeS42YGDe4!0H4E3:2Z>0KeTGeB4eb%c:SeF0eS [0K4'
. L=T4"e1ZTE3GeFGX4_0>Z0U4!e0Fe4cHM4FRGFe F_G?_:F7eGF4eGN83 GK4eFT47O0ARGB0Ve7:_4Re84e_G@C4eG6eW4eFG>3)
" ! ! #!

$$V = \int_{1}^{3.693} \left[(\ln x)^2 \right] dx + \int_{3.693}^{5} \left[(5-x)^2 \right] dx$$

Do not write beyond this border.

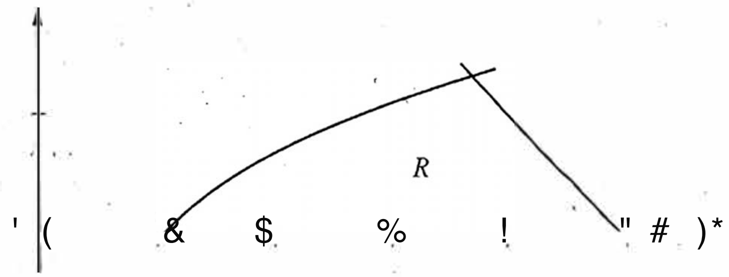
2 - 84e8GKdGFV0e>F4e * e/ "3:_:34Se ":FVGeW GeK47:GFSeG5e4I] 0A0K40)e/ L:T4#e1^TE3GeFGTEFG>_4$0Fe4J] 0T,GF
:F_G?_;F7eGF4eGK6C GK4eFX47F0?Rea 8GS4eFG>TGFe7:_4Re84e 0>4eG5e

$$\text{Area} = \int_{1}^{3.693} [\ln x - k] \, dx + \int_{3.693}^{5} [(5-x) - k] \, dx$$

2 >=+36,A$#B 45?,2* B4 7B%>: %B4 (B
! 2 @B5! 9 B4)B+-; B5! * &B; B00 * ! 1B

" %*%$ * %*' # *$) (*& " ! *

2C1



&" . 3LV V13W14VF38;A@/@L:3V6=9KLVDO02I0@MIAO@232V1UL:3VS0T;KV0@2VL:3V9H0C.KVA6V *V%@V 0@2V +V(V' 0K K:AR@/@L:3V8QB3V01AC3#

0 -;@2VL:3V0I30VA6V

$y = \ln x$ $\quad$ $y = 5 - x$

$x = e^y$ $\quad$ $x = 5 - y$

let $a = 1.30655$

$$\int_0^a \left[(5-y) - (e^y)\right] dy = 2.985$$

3 @?+6:-D?/$E#67B.4*E6;E%A=%E6(E
!5CEB!:?E6)E?,/>E9!*&E6+E22*"1E

, A@N@P5VCHAE>3? V' VB@/@083V$V

Do not write beyond this border.

2 [illegible]

[illegible]

3 [illegible]

$$k = \int_0^a \left[(5-y) - (e^y)\right] dy$$

Question 2

Overview

Students were given the graph of a region R bounded below by the x-axis, on the left by the graph of $y = \ln x$, and on the right by the graph of the line $y = 5 - x$. In part (a) students were asked to find the area of R. This required an appropriate integral setup and evaluation. Students first needed to determine the intersection point, (A, B), of the two curves. The area could then be computed by solving each expression for x in terms of y and evaluating a single integral with respect to y. Alternatively, the area could be computed by evaluating a sum of two integrals with respect to x. In the first case, students would evaluate $\int_0^B \left(5 - y - e^y\right) dy$ and in the second case, $\int_1^A \ln x \, dx + \int_A^5 (5 - x) \, dx$. Part (b) asked for an expression involving one or more integrals that gives the volume of a solid whose base is the region R and whose cross sections perpendicular to the x-axis are squares. Students should have found the cross-sectional area function in terms of x, which is $(\ln x)^2$ on the interval $1 \le x \le A$ and $(5 - x)^2$ on the interval $A \le x \le 5$. These expressions are used as the integrands for two definite integrals with the corresponding endpoints, whose sum provides the desired expression. Part (c) asked for an equation involving one or more integrals whose solution gives the value k for which the line $y = k$ divides the region R into two smaller regions of equal area. Students should have first rewritten the equations for the curves as functions of x in terms of y. Two common solutions were setting the definite integral $\int_0^k \left(5 - y - e^y\right) dy$ equal to half the value of the area computed in part (a), and $\int_0^k \left(5 - y - e^y\right) dy = \int_k^B \left(5 - y - e^y\right) dy$.

Sample: 2A
Score: 9

The student earned all 9 points.

Sample: 2B
Score: 6

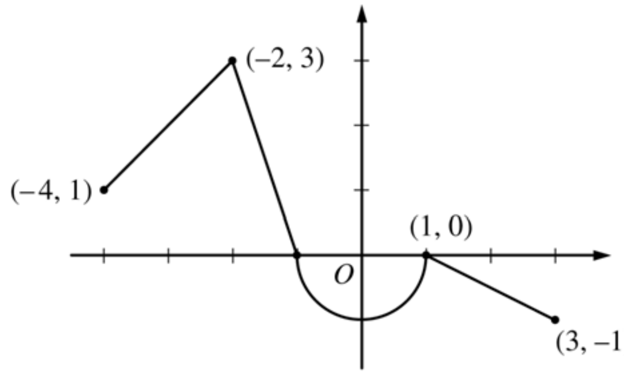
The student earned 6 points: 3 points in part (a), 3 points in part (b), and no points in part (c). In part (a) the student's work is correct. The student works in terms of x in part (a) and correctly sums the two integrals for the area of R. In part (b) the student's work is correct. In part (c) the variable k (or an expression involving k) must be included in the limits of the integrals in order to be eligible for points.

Sample: 2C
Score: 3

The student earned 3 points: 3 points in part (a), no points in part (b), and no points in part (c). In part (a) the student's work is correct. In part (b) the student is not working with the correct cross section. In part (c) the variable k (or an expression involving k) must be included in the limits of the integrals in order to be eligible for points.

Question 3

Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\, dt$.



Graph of f

(a) Find the values of $g(2)$ and $g(-2)$.

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

(a) $g(2) = \int_1^2 f(t)\, dt = -\frac{1}{2}(1)\left(\frac{1}{2}\right) = -\frac{1}{4}$

$g(-2) = \int_1^{-2} f(t)\, dt = -\int_{-2}^{1} f(t)\, dt$

$= -\left(\frac{3}{2} - \frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{3}{2}$

$2 : \begin{cases} 1 : g(2) \\ 1 : g(-2) \end{cases}$

(b) $g'(x) = f(x) \Rightarrow g'(-3) = f(-3) = 2$

$g''(x) = f'(x) \Rightarrow g''(-3) = f'(-3) = 1$

$2 : \begin{cases} 1 : g'(-3) \\ 1 : g''(-3) \end{cases}$

(c) The graph of g has a horizontal tangent line where $g'(x) = f(x) = 0$. This occurs at $x = -1$ and $x = 1$.

$g'(x)$ changes sign from positive to negative at $x = -1$. Therefore, g has a relative maximum at $x = -1$.

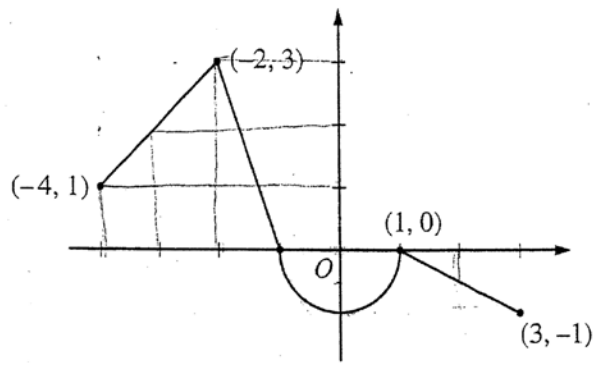
$g'(x)$ does not change sign at $x = 1$. Therefore, g has neither a relative maximum nor a relative minimum at $x = 1$.

$3 : \begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : x = -1 \text{ and } x = 1 \\ 1 : \text{answers with justifications} \end{cases}$

(d) The graph of g has a point of inflection at each of $x = -2$, $x = 0$, and $x = 1$ because $g''(x) = f'(x)$ changes sign at each of these values.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{explanation} \end{cases}$

NO CALCULATOR ALLOWED



Graph of f

3. Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\,dt$.

(a) Find the values of $g(2)$ and $g(-2)$.

$g(2) = \int_1^2 f(t)\,dt$

$g(2) = -\frac{1}{2}(1)(\frac{1}{2})$

$\boxed{g(2) = -\frac{1}{4}}$

$g(-2) = \int_1^{-2} f(t)\,dt$

$g(-2) = \frac{1}{2}\pi(1)^2 - (\frac{1}{2}(1)(3))$

$\boxed{g(-2) = \frac{\pi - 3}{2}}$

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

$g'(x) = f(x)$

$\boxed{g'(-3) = 2}$

$g''(x) = f'(x)$

$\boxed{g''(-3) = 1}$

Do not write beyond this border.

Continue problem 3 on page 15.

NO CALCULATOR ALLOWED

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

$g'(x) = f(x) = 0$

$x = -1, 1$

At $x=-1$ g has a relative maximum because $g'(x)=f(x)$ changes from positive to negative

At $x=1$ g has neither because $g'(x)=f(x)$ does not change sign

(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

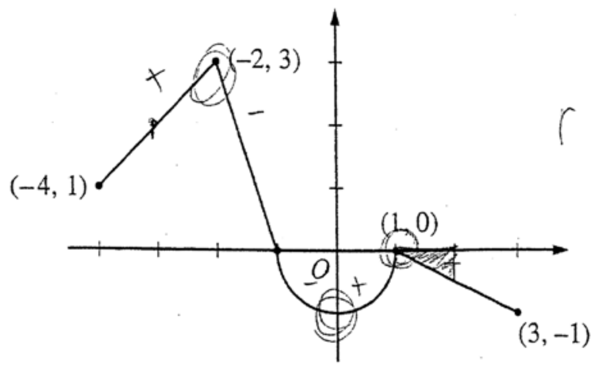
g has inflection points where $g''(x) = f'(x)$ changes sign. This occurs at $x = -2, 0, 1$

Do not write beyond this border.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED



Graph of f

3. Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\,dt$.

(a) Find the values of $g(2)$ and $g(-2)$.

$A = \frac{bh}{2}$

$A = \frac{1(-\frac{1}{2})}{2}$

$A = -\frac{1}{4}$

$g(x) = \int_1^x f(t)\,dt$

$g(2) = \int_1^2 f(t)\,dt$

$g(2) = \frac{1(-.5)}{2}$

$g(2) = -\frac{1}{4}$

$g(-2) = -\int_{-2}^{1} f(t)\,dt$

$A = \frac{bh}{2}$ $A = \pi r^2$

$-\left(\frac{1(3)}{2} - \frac{1}{2}\pi\right)$

$-\frac{3}{2} + \frac{1}{2}\pi$

$g(-2) = \frac{1}{2}\pi - \frac{3}{2}$

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

$m = \frac{\Delta y}{\Delta x}$

$-\frac{2}{2} = -1$

$(-3, 3)$

$g'(x) = f(x)$

$g'(-3) = f(-3)$

$g'(-3) = 3$

$g''(x) = f'(x)$

$g''(-3) = f'(-3)$

$g''(-3) = -1$

Continue problem 3 on page 15.

NO CALCULATOR ALLOWED

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

$g'(x) = 0$ ← horizontal tangent line

$g'(x) = f(x) = 0$

at $x = -1$, $x = 1$

g'	+		−		−
g	inc	−1	dec	1	dec

at $x = -1$, there is a rel. max.
f' changes from + to −

at $x = 1$, there is neither a maximum nor a minimum because the slope remains negative

(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

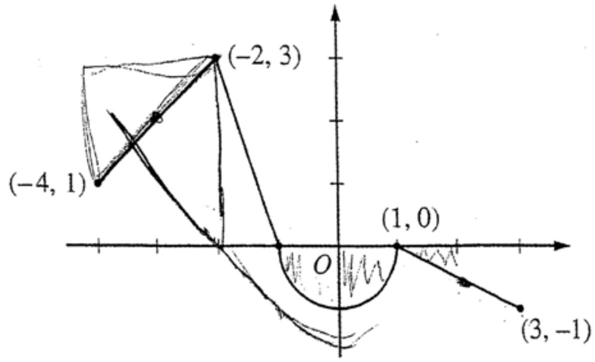
$g''(x) = f'(x)$

g will have a POI where the slope of f changes signs

at $\boxed{x = -2,\ x = 0,\ x = 1}$

GO ON TO THE NEXT PAGE.

3C1



Graph of f

3. Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\,dt$.

(a) Find the values of $g(2)$ and $g(-2)$.

$g'(x) = f(x)$

$g(2) = \pi(1)^2 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{1}{4}\pi + \frac{1}{4}$

$g(-2) = -\frac{1}{4}(1)^2\pi + \frac{1}{2} \cdot 3 \cdot 1 = -\frac{1}{4}\pi + \frac{3}{2}$

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

$g'(x) = f(x)$

$g'(-3) = f(-3) = 2$

$g''(x) = f'(x)$

$g''(-3) = f'(-3) = 1$

Do not write beyond this border.

Continue problem 3 on page 15.

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

$g'(x)=0$ $g'(x)=f(x)=0$

$g'(x)=f(x)=0$ at $x=-1$

The point at $x=-1$ is a maximum because the graph of $f(x)|g'(x)$ transitions from positive to negative at this point. This means that on the original graph ($g(x)$) the graph is moving from increasing to decreasing at this point, which is representative of a maximum.

(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

$g''(x)=f'(x)$

There is a point of inflection at $x=-2,3$ because on the $f(x)|g'(x)$ graph they are presented as extrema. Extrema on a first derivative graph represents a point of inflection on the original graph. Therefore the extrema on the graph given, $x=-2,3$, are points of inflection on $g(x)$.

GO ON TO THE NEXT PAGE.

Do not write beyond this border.

Question 3

Overview

This problem described a function f that is defined and continuous on the interval $[-4, 3]$. The graph of f on $[-4, 3]$ is given and consists of three line segments and a semicircle. The function g is defined by $g(x) = \int_1^x f(t)\,dt$. Part (a) asked for the values of $g(2)$ and $g(-2)$. These values are given by $\int_1^2 f(t)\,dt$ and $\int_1^{-2} f(t)\,dt$, respectively, and are computed using geometry and a property of definite integrals. Part (b) asked for the values of $g'(-3)$ and $g''(-3)$, provided they exist. Students should have applied the Fundamental Theorem of Calculus to determine that $g'(-3) = f(-3)$ and $g''(-3) = f'(-3)$. Students should have used the graph provided to determine the value of f and the slope of f at the point where $x = -3$. Part (c) asked for the x-coordinate of each point where the graph of g has a horizontal tangent line. Students were then asked to classify each of these points as the location of a relative minimum, relative maximum, or neither, with justification. Students should have recognized that horizontal tangent lines for g occur where the derivative of g takes on the value 0. These values can be read from the graph. Students should have applied a sign analysis to f in order to classify these critical points. Part (d) asked for the x-coordinates of points of inflection for the graph of g on the interval $-4 < x < 3$. Students should have reasoned graphically that these occur where f changes from increasing to decreasing, or vice versa.

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

The student earned 6 points: 2 points in part (a), no points in part (b), 2 points in part (c), and 2 points in part (d). In parts (a) and (d) the student's work is correct. In part (b) the student does not supply the correct values. In part (c) the student correctly considers $g'(x) = 0$ and identifies the correct x-values. The student does not give a correct justification for $x = -1$, confusing f' with g', and does not specify which function's slope is intended in the justification for $x = 1$.

Sample: 3C
Score: 3

The student earned 3 points: no points in part (a), 2 points in part (b), 1 point in part (c), and no points in part (d). In part (a) the student does not supply the correct values. In part (b) the student's work is correct. In part (c) the student considers $g'(x) = 0$, so the first point was earned. The student identifies only one of the x-values, so the second point was not earned. The student is not eligible for the third point. In part (d) the student does not identify the correct x-values.

Question 4

The function f is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \le x \le 5$.

(a) Find $f'(x)$.

(b) Write an equation for the line tangent to the graph of f at $x = -3$.

(c) Let g be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x + 7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x = -3$? Use the definition of continuity to explain your answer.

(d) Find the value of $\int_0^5 x\sqrt{25 - x^2}\, dx$.

Solution	Scoring
(a) $f'(x) = \frac{1}{2}\left(25 - x^2\right)^{-1/2}(-2x) = \frac{-x}{\sqrt{25 - x^2}}, \quad -5 < x < 5$	2 : $f'(x)$
(b) $f'(-3) = \frac{3}{\sqrt{25 - 9}} = \frac{3}{4}$ $f(-3) = \sqrt{25 - 9} = 4$ An equation for the tangent line is $y = 4 + \frac{3}{4}(x + 3)$.	2 : $\begin{cases} 1 : f'(-3) \\ 1 : \text{answer} \end{cases}$
(c) $\lim\limits_{x \to -3^-} g(x) = \lim\limits_{x \to -3^-} f(x) = \lim\limits_{x \to -3^-} \sqrt{25 - x^2} = 4$ $\lim\limits_{x \to -3^+} g(x) = \lim\limits_{x \to -3^+} (x + 7) = 4$ Therefore, $\lim\limits_{x \to -3} g(x) = 4$. $g(-3) = f(-3) = 4$ So, $\lim\limits_{x \to -3} g(x) = g(-3)$. Therefore, g is continuous at $x = -3$.	2 : $\begin{cases} 1 : \text{considers one-sided limits} \\ 1 : \text{answer with explanation} \end{cases}$
(d) Let $u = 25 - x^2 \Rightarrow du = -2x\,dx$ $\int_0^5 x\sqrt{25 - x^2}\, dx = -\frac{1}{2}\int_{25}^{0} \sqrt{u}\, du$ $= \left[-\frac{1}{2} \cdot \frac{2}{3} u^{3/2}\right]_{u=25}^{u=0}$ $= -\frac{1}{3}(0 - 125) = \frac{125}{3}$	3 : $\begin{cases} 2 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

 4A₁

NO CALCULATOR ALLOWED

4. The function f is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \le x \le 5$.

(a) Find $f'(x)$. $= (25-x^2)^{1/2}$

$$f'(x) = \frac{1 \cdot -2x}{2\sqrt{25-x^2}} = \boxed{\frac{-x}{\sqrt{25-x^2}}}$$

(b) Write an equation for the line tangent to the graph of f at $x = -3$.

$$f'(-3) = \frac{3}{\sqrt{25-9}} = \frac{3}{\sqrt{16}} = \frac{3}{4} \qquad f(-3) = \sqrt{25-9} = \sqrt{16} = 4$$

$$\boxed{y = \frac{3}{4}(x+3)+4}$$

Do not write beyond this border.

Do not write beyond this border.

Continue problem 4 on page 17.

4A2

NO CALCULATOR ALLOWED

(c) Let g be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x+7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x = -3$? Use the definition of continuity to explain your answer.

$\lim_{x \to -3^-} g(x) = f(-3) = 4$

$\lim_{x \to -3^+} g(x) = -3+7 = 4$

therefore $\lim_{x \to -3} g(x) = 4$

$g(-3) = f(-3) = 4$

$\lim_{x \to -3} g(x)$ exists and is equal to $g(-3)$

g is continuous at $x = -3$

(d) Find the value of $\int_0^5 x\sqrt{25 - x^2}\, dx$.

$u = 25 - x^2$

$-\frac{1}{2} du = x\, dx$

$-\frac{1}{2}\int_{25}^{0} u^{1/2}\, du = \frac{1}{2}\int_{0}^{25} u^{1/2}\, du = \frac{1}{2} \cdot \frac{2u^{3/2}}{3} \Big|_0^{25}$

$= \frac{1}{3} \cdot 25^{3/2} = \frac{125}{3}$

Do not write beyond this border.

Unauthorized copying or reuse of any part of this page is illegal.

GO ON TO THE NEXT PAGE.

4. The function f is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \le x \le 5$.

(a) Find $f'(x)$.

$f'(x) = \frac{1}{2}(25 - x^2)^{-\frac{1}{2}}(-2x)$

$f'(x) = \dfrac{-x}{\sqrt{25 - x^2}}$

Do not write beyond this border.

(b) Write an equation for the line tangent to the graph of f at $x = -3$.

$f(-3) = \sqrt{16} = 4$

$f'(-3) = \dfrac{3}{4}$

$y - 4 = \frac{3}{4}(x + 3)$

Do not write beyond this border.

Continue problem 4 on page 17.

4B2

NO CALCULATOR ALLOWED

(c) Let g be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x+7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x = -3$? Use the definition of continuity to explain your answer.

$f(-3) = 4$ yes, $g(x)$ is continuous at $x = -3$

$-3 + 7 = 4$

In order for a fct to be continuous, the $\lim_{x \to h^+}$ must equal $\lim_{x \to h^-}$ and the actual value of the fct must be equivalent to the value that the fct approaches from both the left and the right at y=h

(d) Find the value of $\int_0^5 x\sqrt{25 - x^2}\, dx$.

$u = 25 - x^2$

$dx = \frac{du}{2x}$

$\frac{1}{2}\int_{25}^{0} \sqrt{u}\, du = -\frac{1}{2}\int_0^{25} \sqrt{u}\, du = -\frac{1}{2}\left(\frac{2}{3}\right)(u)^{\frac{3}{2}}\Big|_0^{25}$

$= -\frac{1}{3}(25)^{\frac{3}{2}} + \frac{1}{3}(0)^{\frac{3}{2}} = -\frac{125}{3}$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

 4C,

4. The function f is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \le x \le 5$.

(a) Find $f'(x)$.

$f'(x) = (25 - x^2)^{1/2}$

$= 1/2 (25 - x^2)^{-1/2}(-2x)$

$-x(25 - x^2)^{-1/2}$

$\frac{-x}{\sqrt{25 - x^2}}$

(b) Write an equation for the line tangent to the graph of f at $x = -3$.

$\frac{-3}{\sqrt{25 - (-3)^2}} = \frac{-3}{\sqrt{25 - 9}} = \frac{-3}{4}$

$y - 4 = \frac{-3}{4}(x + 3)$

Do not write beyond this border.

Continue problem 4 on page 17.

NO CALCULATOR ALLOWED

(c) Let g be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x+7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x = -3$? Use the definition of continuity to explain your answer.

$f(x) = x+7$

$f(x) = 3+7$

$f(x) = 10$

No g is not continuous at x = 3.

(d) Find the value of $\int_0^5 x\sqrt{25-x^2}\,dx$.

$u = 25 - x^2$

$du = -2x$

$\cdot \frac{1}{2}\int_0^5 x\sqrt{u}\,du$

$\frac{1}{2}x^2\left(\frac{2}{3}u^{3/2}\right)\Big|_0^5$

$\left(-\frac{1}{2}\right)\frac{1}{2}x^2\left(\frac{2}{3}(25-x^2)^{3/2}\right)\Big|_0^5 =$

Do not write beyond this border.

Unauthorized copying or reuse of any part of this page is illegal.

GO ON TO THE NEXT PAGE.

Question 4

Overview

This problem presented a function f defined by $f(x) = \sqrt{25 - x^2}$ on the interval $-5 \le x \le 5.$ In part (a) students were asked to find the derivative $f'(x).$ This involved correctly applying the chain rule to determine the symbolic derivative of f. Part (b) asked for an equation of the line tangent to the graph of f at the point where $x = -3.$ Students needed to find the derivative at this point to determine the slope of the tangent line, the y-coordinate of the graph of f at this point, and then combine this information to provide an equation for the line. Part (c) presented a piecewise-defined function g that is equal to f on the interval $-5 \le x \le -3$ and to $x + 7$ on the interval $-3 < x \le 5.$ Students were asked to use the definition of continuity to determine whether g is continuous at $x = -3.$ Students should have evaluated the left-hand and right-hand limits as x approaches $-3,$ and observed that these are the same and equal to the function value at that point. Part (d) asked students to evaluate the definite integral $\int_0^5 x\sqrt{25 - x^2}\, dx,$ which can be done using the substitution $u = 25 - x^2.$

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: 2 points in part (a), 2 points in part (b), no points in part (c), and 2 points in part (d). In parts (a) and (b) the student's work is correct. In part (c) the student's work is not sufficient for any points. In part (d) the student earned 1 of the 2 antiderivative points owing to a sign error. The student evaluates the definite integral in a manner consistent with the sign error and earned the answer point.

Sample: 4C
Score: 3

The student earned 3 points: 2 points in part (a), 1 point in part (b), no points in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student evaluates f' incorrectly but uses this value as the slope along with the point $(-3, 4)$ to write an equation of the tangent line. In parts (c) and (d) the student's work is incorrect.

AP® CALCULUS AB
2012 SCORING GUIDELINES

Question 5

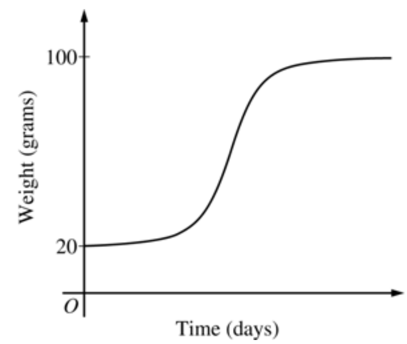
The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0,$ when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20.$

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



(c) Use separation of variables to find $y = B(t),$ the particular solution to the differential equation with initial condition $B(0) = 20.$

(a) $\left.\frac{dB}{dt}\right|_{B=40} = \frac{1}{5}(60) = 12$

$\left.\frac{dB}{dt}\right|_{B=70} = \frac{1}{5}(30) = 6$

Because $\left.\frac{dB}{dt}\right|_{B=40} > \left.\frac{dB}{dt}\right|_{B=70},$ the bird is gaining weight faster when it weighs 40 grams.

$2 : \begin{cases} 1 : \text{uses } \frac{dB}{dt} \\ 1 : \text{answer with reason} \end{cases}$

(b) $\frac{d^2B}{dt^2} = -\frac{1}{5}\frac{dB}{dt} = -\frac{1}{5}\cdot\frac{1}{5}(100 - B) = -\frac{1}{25}(100 - B)$

Therefore, the graph of B is concave down for $20 \le B < 100.$ A portion of the given graph is concave up.

$2 : \begin{cases} 1 : \frac{d^2B}{dt^2} \text{ in terms of } B \\ 1 : \text{explanation} \end{cases}$

(c) $\frac{dB}{dt} = \frac{1}{5}(100 - B)$

$\int \frac{1}{100 - B}\, dB = \int \frac{1}{5}\, dt$

$-\ln|100 - B| = \frac{1}{5}t + C$

Because $20 \le B < 100,$ $|100 - B| = 100 - B.$

$-\ln(100 - 20) = \frac{1}{5}(0) + C \Rightarrow -\ln(80) = C$

$100 - B = 80e^{-t/5}$

$B(t) = 100 - 80e^{-t/5}, \quad t \ge 0$

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } B \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

5A1

Do not write beyond this border.

5. The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

when it is 40 grams: $\frac{dB}{dt} = \frac{1}{5}(100-40) = 12$ g/day

when it is 70 grams: $\frac{dB}{dt} = \frac{1}{5}(100-70) = 6$ g/day

so the bird is gaining weight faster when it weighs 40 grams.

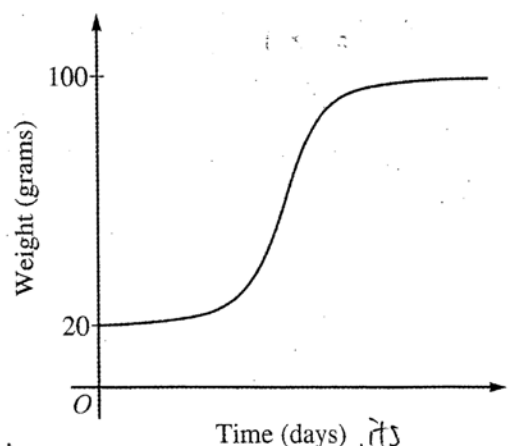
(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.

$\frac{dB}{dt} = 20 - \frac{1}{5}B$

$\frac{d^2B}{dt^2} = -\frac{1}{5} \cdot \frac{dB}{dt}$

$= -\frac{1}{5}(20 - \frac{1}{5}B)$

$= \frac{1}{25}B - 4$



$\frac{1}{25}B - 4 > 0$

$B > 100$

so, the graph cannot be concave up when its weight is below 100 g

Continue problem 5 on page 19.

NO CALCULATOR ALLOWED

5A₂

(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

$$\frac{dB}{dt} = \frac{1}{5}(100 - B)$$

$$\frac{1}{\frac{1}{5}(100-B)}\, dB = dt$$

$$\frac{5}{100-B}\, dB = dt$$

$$\int \frac{5}{100-B}\, dB = \int 1\, dt$$

$$-5\ln[100-B] = t + c$$

$$\ln(100-B) = -\frac{1}{5}(t+c)$$

$$100 - B = e^{-\frac{1}{5}(t+c)}$$

$$B = 100 - e^{-\frac{1}{5}(t+c)}$$

$$20 = 100 - e^{-\frac{1}{5}c}$$

$$e^{-\frac{1}{5}c} = 80$$

$$-\frac{1}{5}c = \ln 80$$

$$c = -5\ln 80$$

$$\therefore B = 100 - e^{-\frac{1}{5}(t - 5\ln 80)}$$

Do not write beyond this border.

Unauthorized copying or reuse of any part of this page is illegal.

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

5B₁

5. The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

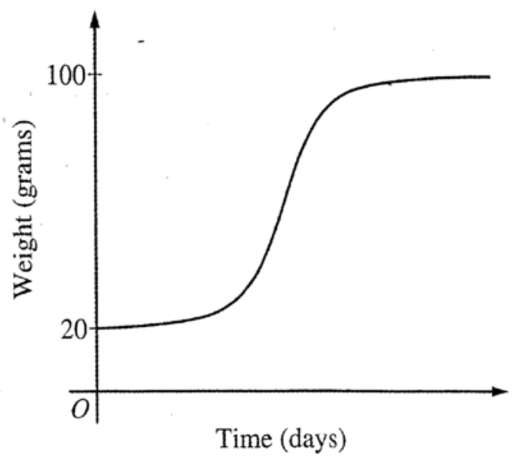
(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

Weight = 40 ⇒ $\frac{dB}{dt} = \frac{1}{5}(100-40) = \frac{60}{5}$ gram/day

Weight = 70 ⇒ $\frac{dB}{dt} = \frac{1}{5}(100-70) = \frac{30}{5}$ gram/day

$\frac{60}{5} > \frac{30}{5}$ ⇒ at weight = 40 gram, the rate of change of Bird weight is faster.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



$\frac{d^2B}{dt^2} = \frac{1}{5}\left(0 - \frac{dB}{dt}\right) = -\frac{1}{5}\frac{dB}{dt}$, $\frac{d^2B}{dt^2}$ is negative

⇒ the graph of B has to be concave down all the times

Continue problem 5 on page 19.

NO CALCULATOR ALLOWED

5B2

(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

$$\frac{dB}{dt} = \frac{1}{5}(100 - B)$$

$$\int \frac{dB}{100-B} = \int \frac{1}{5}\, dt$$

$$\ln(100 - B) = \frac{t}{5} + C$$

$$100 - B = Ce^{t/5}$$

$B(0) = 20 \Rightarrow 100 - 20 = Ce^{0} \Rightarrow C = 80$

$\Rightarrow$ partialar solution: $100 - B = 80e^{t/5}$

Do not write beyond this border.

Do not write beyond this border.

Unauthorized copying...

5C1

5. The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

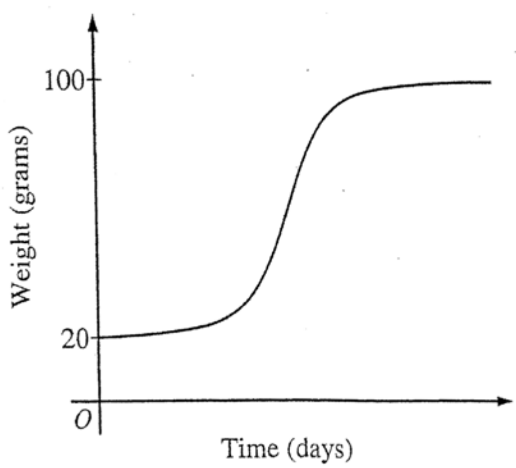
(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

$\frac{1}{5}(100-40)$ $\frac{60}{5}$ 12

$\frac{1}{5}(100-70)$ $\frac{30}{5}$ 6

gains weight faster when it weighs 40 grams because its growing at twice the rate it is when its 70 grams.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



$20 - \frac{B}{5}$

$\frac{d^2B}{dt^2} \frac{d}{dB} = -\frac{1}{5}$

Because $\frac{d^2B}{dt^2}$ is $-\frac{1}{5}$, this can't resemble the following because the concavity isn't negative.

Do not write beyond this border.

Continue problem 5 on page 19.

5C2

(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

$$\int \frac{1}{5}\,dt = \int \frac{dB}{100-B}$$

$$C + \frac{1}{5}t = -\frac{1}{2}(100-B)^{-2}$$

$$\frac{1}{5}t + C = \frac{-1}{2(100-B)^2}$$

$$-2\left(\frac{1}{5}t + C\right) = \frac{1}{100-B^2}$$

$$100 - B^2 = \frac{1}{-2\left(\frac{1}{5}t + C\right)}$$

$$100 + \frac{1}{+2\left(\frac{1}{5}t + C\right)} = B^2$$

$$\sqrt{100 + \frac{1}{2\left(\frac{1}{5}t + C\right)}} = B$$

$$\sqrt{100 + \frac{1}{2(C)}} = 20$$

$$\begin{array}{r} 20 \\ 20 \\ \hline 400 \end{array}$$

$$\frac{1}{2C} = 300$$

$$\frac{1}{300} = 2C$$

$$\frac{\frac{1}{300}}{2}$$

$$C = \frac{2}{300}$$

$$B = \sqrt{100 + \frac{1}{2\left(\frac{1}{5}t + \frac{2}{300}\right)}}$$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 5

Overview

The context of this problem is weight gain of a baby bird. At time $t = 0,$ when the bird is first weighed, its weight is 20 grams. A function B modeling the weight of the bird satisfies $\frac{dB}{dt} = \frac{1}{5}(100 - B),$ where t is measured in days since the bird was first weighed. Part (a) asked whether the bird is gaining weight faster when it weighs 40 grams or when it weighs 70 grams. Students had to evaluate and compare $\frac{dB}{dt}$ for these two values of B. Part (b) asked for $\frac{d^2B}{dt^2}$ in terms of B. Students should have used a sign analysis of the second derivative to explain why the graph of B cannot resemble the given graph. Part (c) asked students to use separation of variables to solve the initial value problem $\frac{dB}{dt} = \frac{1}{5}(100 - B)$ with $B(0) = 20$ to find $B(t)$.

Sample: 5A
Score: 9

The student earned all 9 points. Note that in part (c) the student does not need absolute value on the fifth line because $B(0) = 20.$

Sample: 5B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), and 3 points in part (c). In part (a) the student's work is correct. In part (b) the first point was not earned because the student does not present $\frac{d^2B}{dt^2}$ in terms of B. The student's correct appeal to the chain rule and correct explanation earned the second point. In part (c) the student earned the first point with a correct separation on the second line. The second point was not earned because the student's antiderivative on the left-hand side on the third line is incorrect. (The antiderivative should be $-\ln(100 - B),$ with no absolute value needed.) A student who did not earn the second point is not eligible for the fifth point. The student earned the third point on the third line and the fourth point on the fifth line for correctly substituting 0 for t and 20 for B.

Sample: 5C
Score: 3

The student earned 3 points: 2 points in part (a), no points in part (b), and 1 point in part (c). In part (a) the student's work is correct. In part (b) the student makes a chain rule error and did not earn the first point. The student is not eligible for the second point in part (b). In part (c) the student presents a correct separation on the first line and earned the first point. The student's incorrect B-antiderivative makes the student ineligible for any additional points in part (c).

Question 6

For $0 \le t \le 12$, a particle moves along the x-axis. The velocity of the particle at time t is given by $v(t) = \cos\left(\frac{\pi}{6}t\right)$. The particle is at position $x = -2$ at time $t = 0$.

(a) For $0 \le t \le 12$, when is the particle moving to the left?

(b) Write, but do not evaluate, an integral expression that gives the total distance traveled by the particle from time $t = 0$ to time $t = 6$.

(c) Find the acceleration of the particle at time t. Is the speed of the particle increasing, decreasing, or neither at time $t = 4$? Explain your reasoning.

(d) Find the position of the particle at time $t = 4$.

(a) $v(t) = \cos\left(\frac{\pi}{6}t\right) = 0 \quad \Rightarrow \quad t = 3,\ 9$

The particle is moving to the left when $v(t) < 0$.
This occurs when $3 < t < 9$.

2 : { 1 : considers $v(t) = 0$; 1 : interval }

(b) $\int_0^6 |v(t)|\, dt$

1 : answer

(c) $a(t) = -\frac{\pi}{6}\sin\left(\frac{\pi}{6}t\right)$

$a(4) = -\frac{\pi}{6}\sin\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}\pi}{12} < 0$

$v(4) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} < 0$

The speed is increasing at time $t = 4$, because velocity and acceleration have the same sign.

3 : { 1 : $a(t)$; 2 : conclusion with reason }

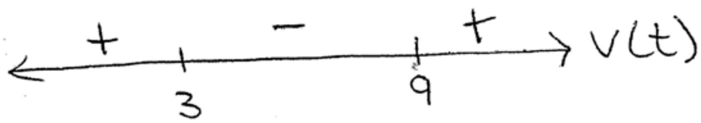
(d)
$$
\begin{aligned}
x(4) &= -2 + \int_0^4 \cos\left(\frac{\pi}{6}t\right) dt \\
&= -2 + \left[\frac{6}{\pi}\sin\left(\frac{\pi}{6}t\right)\right]_0^4 \\
&= -2 + \frac{6}{\pi}\left[\sin\left(\frac{2\pi}{3}\right) - 0\right] \\
&= -2 + \frac{6}{\pi}\cdot\frac{\sqrt{3}}{2} = -2 + \frac{3\sqrt{3}}{\pi}
\end{aligned}
$$

3 : { 1 : antiderivative; 1 : uses initial condition; 1 : answer }

6. For $0 \le t \le 12$, a particle moves along the x-axis. The velocity of the particle at time t is given by $v(t) = \cos\left(\frac{\pi}{6}t\right)$. The particle is at position $x = -2$ at time $t = 0$. $x(0) = -2$

(a) For $0 \le t \le 12$, when is the particle moving to the left?

$\cos(\frac{\pi}{6}t) = 0$ when $x = 3, x = 9$



the particle is moving lest on (3, 9)

(b) Write, but do not evaluate, an integral expression that gives the total distance traveled by the particle from time $t = 0$ to time $t = 6$.

$D = \int_0^6 \left|\cos(\frac{\pi}{6}t)\right| dt$

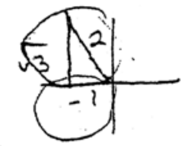
Do not write beyond this border.

Continue problem 6 on page 21.

NO CALCULATOR ALLOWED

(c) Find the acceleration of the particle at time t. Is the speed of the particle increasing, decreasing, or neither at time $t = 4$? Explain your reasoning.

$a(t) = v'(t) = -\sin\left(\frac{\pi}{6}t\right) \cdot \frac{\pi}{6} = \boxed{-\frac{\pi}{6}\sin\left(\frac{\pi}{6}t\right)}$



$a(4) = -\frac{\pi}{6}\sin\left(\frac{2\pi}{3}\right) = -\frac{\pi}{6}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi\sqrt{3}}{12}$

$v(4) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

The speed is increasing because a(t) and v(t) have the same sign (-) at t=4

(d) Find the position of the particle at time $t = 4$.

$x(t) = \int \cos\left(\frac{\pi}{6}t\right) dt \quad u = \frac{\pi}{6}t \quad du = \frac{\pi}{6}dt$

$x(t) = \frac{6}{\pi}\int \cos u \, du$

$x(t) = \frac{6}{\pi}\sin\left(\frac{\pi}{6}t\right) + C$

$\frac{6}{\pi}\sin\left(\frac{\pi}{6}t\right) + C = -2$ when $x = 0$

$\frac{6}{\pi}\sin(0) + C = -2$

$C = -2$

$x(t) = \frac{6}{\pi}\sin\left(\frac{\pi}{6}t\right) - 2$

$x(4) = \frac{6}{\pi}\sin\left(\frac{2\pi}{3}\right) - 2$

$x(4) = \frac{6}{\pi}\left(\frac{\sqrt{3}}{2}\right) - 2$

$\boxed{x(4) = \frac{3\sqrt{3}}{\pi} - 2}$

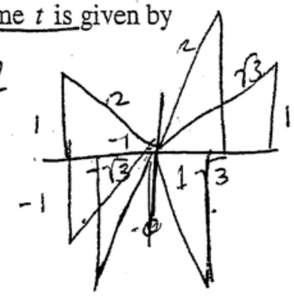
Do not write beyond this border.

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

6. For $0 \le t \le 12$, a particle moves along the x-axis. The velocity of the particle at time t is given by $v(t) = \cos\left(\frac{\pi}{6}t\right)$. The particle is at position $x = -2$ at time $t = 0$.

$X(0) = -2$



(a) For $0 \le t \le 12$, when is the particle moving to the left?

$t = 0 \rightarrow 1$

$t = 1 \rightarrow \sqrt{3}/2$

$t = 2 \rightarrow 1/2$

$t = 3 = 0$

$t = 4 = -\frac{1}{2}$

$t = 5 = -\frac{\sqrt{3}}{2}$

$t = 6 = -1$

$t = 7 = -\frac{\sqrt{3}}{2}$

$t = 8 = -\frac{1}{2}$

$t = 9 = 0$

$t = 10 =$ ←

$(4, 8)$

(b) Write, but do not evaluate, an integral expression that gives the total distance traveled by the particle from time $t = 0$ to time $t = 6$.

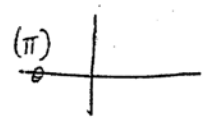
$x(t) = \int_0^6 |v(t)|\,dt \quad u = \frac{\pi}{6}t$

$du = \frac{\pi}{6}\,dt$

$\frac{6}{\pi}\int \cos(u)\,du$

$\frac{6}{\pi}\left[\sin\left(\frac{\pi}{6}\right)\right]_0^6$

$\frac{6}{\pi}\left[\sin\left(6 \cdot \frac{\pi}{6}\right) - \sin(0)\right]$

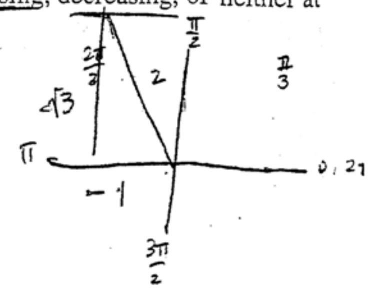


$X(t) = \int_0^6 |v(t)|\,dt - 2$

Continue problem 6 on page 21.

Do not write beyond this border.

(c) Find the acceleration of the particle at time t. Is the speed of the particle increasing, decreasing, or neither at time $t = 4$? Explain your reasoning.



$a(t) = v'(t) = -\sin\left(\frac{\pi}{6}t\right)\cdot\frac{\pi}{6}$

$a(t) = -\frac{\pi}{6}\sin\left(\frac{\pi}{6}t\right)$

$a(4) = -\frac{\pi}{6}\sin\left(\frac{\pi}{6}\cdot 4\right) = -\frac{\pi}{6}\sin\left(\frac{2\pi}{3}\right)$

$-\frac{\pi}{6}\cdot\frac{\sqrt{3}}{2} = \boxed{-\frac{\pi\sqrt{3}}{12}} = a(t)$ at $t=4$

$v(4) = \cos\frac{\pi}{6}\cdot 4 = \cos\frac{2\pi}{3} = -\frac{1}{2}$

The particle is increasing because velocity and acceleration have the same sign.

$\frac{2\pi}{3} \cdot \frac{180}{\pi}$

(d) Find the position of the particle at time $t = 4$.

$x(t) = \int_0^4 v(t)\,dt - 2$

$\frac{6}{\pi}\int \cos(u)\,du - 2$

$\frac{6}{\pi}\left[\sin\left(\frac{\pi}{6}t\right)\right]_0^4 - 2$

$\frac{6}{\pi}\left(\frac{\sqrt{3}}{2} - 0\right) - 2$

$= \boxed{\frac{3\sqrt{3}}{\pi} - 2}$

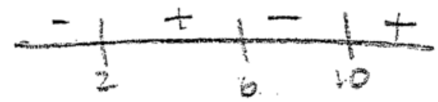
Do not write beyond this border.

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

6. For $0 \le t \le 12$, a particle moves along the x-axis. The velocity of the particle at time t is given by $v(t) = \cos\left(\frac{\pi}{6}t\right)$. The particle is at position $x = -2$ at time $t = 0$.

(a) For $0 \le t \le 12$, when is the particle moving to the left?



$(0, 2),\ (6, 10)$

(b) Write, but do not evaluate, an integral expression that gives the total distance traveled by the particle from time $t = 0$ to time $t = 6$.

total distance $= \int_0^6 |v(t)|\,dt$

Do not write beyond this border.

Continue problem 6 on page 21.

6C₂

NO CALCULATOR ALLOWED

(c) Find the acceleration of the particle at time t. Is the speed of the particle increasing, decreasing, or neither at time $t = 4$? Explain your reasoning.

$A(t) = v'(t)$

$-\sin\left(\frac{\pi}{6}t\right) \cdot \frac{\pi}{6} = -\frac{\pi}{6}\sin\left(\frac{\pi}{6}t\right)$

$A(4) = -\frac{\pi}{6}\sin\left(\frac{\pi}{6}\cdot 4\right) = \frac{\sqrt{3}}{2} \cdot -\frac{\pi}{6} = \text{neg}$

∴ the speed is decreasing at $t=4$ because the acceleration is negative

(d) Find the position of the particle at time $t = 4$.

$P(t) = \int v(t)\,dt$

$\int \cos\left(\frac{\pi}{6}t\right)dt$ $\quad u = \frac{\pi}{6}t$ $\quad du = \frac{\pi}{6}dt$

$6\pi \int \cos u\,du$

$6\pi \sin u + C$

$P(t) = 6\pi \sin\left(\frac{\pi}{6}t\right) + C$

$-2 = 6\pi \sin\left(\frac{\pi}{6}(0)\right) + C$

$-2 = 0 + C$

$C = -2$

$P(4) = 6\pi \sin\left(\frac{\pi}{6}(4)\right) - 2$

$6\pi \cdot \frac{\sqrt{3}}{2}$

$P(4) = 3\pi\sqrt{3} - 2$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 6

Overview

This problem presented students with a particle in rectilinear motion during the time interval $0 \le t \le 12$. The particle's position at time $t = 0$ is given, and the velocity $v(t)$ is provided. Part (a) asked students to determine the times when the particle is moving to the left, which they should have done by considering the sign of the given velocity function. In part (b) students were asked to provide an integral expression for the total distance traveled by the particle from time $t = 0$ to time $t = 6$, which they should have recognized as given by the definite integral of $|v(t)|$ over the given time interval. Part (c) asked for the acceleration at time t. Students should have recognized that the acceleration $a(t)$ is the derivative of the velocity function. Students should have provided a symbolic derivative for the given velocity function, correctly using the chain rule. Students were then asked whether the speed of the particle is increasing, decreasing, or neither at time $t = 4$. Students should have evaluated both the velocity and the acceleration functions at time $t = 4$. Because $v(4) < 0$ and $a(4) < 0$, the speed of the particle is increasing. Part (d) asked students to find the position of the particle at time $t = 4$. This is calculated using the expression $x(4) = x(0) + \int_0^4 v(t)\,dt$.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: no points in part (a), no points in part (b), 3 points in part (c), and 3 points in part (d). In part (a) the student does not consider $v(t) = 0$ and gives an incorrect interval. In part (b) the student subtracts 2 from the correct expression. In parts (c) and (d) the student's work is correct. Because "the speed of the particle" is included in the question for part (c), the student's "The particle is increasing" is acceptable.

Sample: 6C
Score: 3

The student earned 3 points: no points in part (a), 1 point in part (b), 1 point in part (c), and 1 point in part (d). In part (a) the student does not consider $v(t) = 0$ and gives an incorrect interval. In part (b) the student's work is correct. In part (c) the student has a correct expression for acceleration, so the first point was earned. No additional points in part (c) were earned because the student uses an argument based on $a(4)$ with no mention of $v(4)$. In part (d) the student presents an incorrect antiderivative. The student uses the incorrect antiderivative with the initial condition and earned the second point. The student is not eligible for the answer point.